

Exercices Fractions rationnelles

①

1) Décomposer en éléments simples dans $\mathbb{C}(X)$ puis dans $\mathbb{R}(X)$
la fraction: $F = \frac{1}{(X^2+1)(X^2+X+1)}$

Décomposition dans $\mathbb{C}(X)$

$$F = \frac{1}{(X-i)(X+i)(X-j)(X-\bar{j})}$$

$$= \frac{a}{X-i} + \frac{b}{X+i} + \frac{c}{X-j} + \frac{d}{X-\bar{j}} \quad \text{avec } a, b, c, d \in \mathbb{C}$$

$$a = F(X)(X-i) \Big|_{X=i} = \frac{1}{(X+i)(X^2+X+1)} \Big|_{X=i} = \frac{1}{2i(i^2+i+1)} = -\frac{1}{2}$$

$$b = F(X)(X+i) \Big|_{X=-i} = \frac{1}{(X-i)(X^2+X+1)} \Big|_{X=-i} = -\frac{1}{2}$$

$$c = F(X)(X-j) \Big|_{X=j} = \frac{1}{(X^2+1)(X-\bar{j})} \Big|_{X=j} = \frac{1}{(j^2+1)(j-\bar{j})}$$

Or $1+j+j^2=0 \Rightarrow 1+j^2=-j$ et $\bar{j}=j^2$ d'où $c = \frac{1}{j(j-j^2)} = \frac{1}{j^2-j^3}$

$$= \frac{1}{j^2-1} \quad (\text{car } j^3=1) \quad \text{Donc } c = \frac{1}{1-j^2} = \frac{1}{1-\bar{j}} \quad ; \text{ Or } j = \frac{-1+i\sqrt{3}}{2}$$

donc $\bar{j} = \frac{-1-i\sqrt{3}}{2} \Rightarrow 1-\bar{j} = 1 + \frac{1+i\sqrt{3}}{2} = \frac{3+i\sqrt{3}}{2} \Rightarrow c = \frac{2}{3+i\sqrt{3}}$

$$c = \frac{2(3-i\sqrt{3})}{(3+i\sqrt{3})(3-i\sqrt{3})} = \frac{2(3-i\sqrt{3})}{9+3} = \frac{3-i\sqrt{3}}{6} ; \text{ de même, on obtient}$$

$$d = F(X)(X-\bar{j}) \Big|_{X=\bar{j}} = \frac{3+i\sqrt{3}}{6}$$

Remarque:
 $b = \bar{a}$ et $d = \bar{c}$

D'où $F(X) = -\frac{1}{2(X-i)} - \frac{1}{2(X+i)} + \frac{3-i\sqrt{3}}{6(X-j)} + \frac{3+i\sqrt{3}}{6(X-\bar{j})}$

Décomposition dans $\mathbb{R}(X)$

On a $-\frac{1}{2(X-i)} - \frac{1}{2(X+i)} = -\frac{1}{2} \left(\frac{X+i+X-i}{(X-i)(X+i)} \right) = -\frac{1}{2} \frac{2X}{X^2+1} = -\frac{X}{X^2+1}$

②

$$\frac{3-i\sqrt{3}}{6(x-j)} + \frac{3+i\sqrt{3}}{6(x-\bar{j})} = \frac{(3-i\sqrt{3})(x-\bar{j}) + (3+i\sqrt{3})(x-j)}{6(x-j)(x-\bar{j})}$$

$$= \frac{3x - 3\bar{j} - i\sqrt{3}x + i\sqrt{3}\bar{j} + 3x - 3j + i\sqrt{3}x - i\sqrt{3}j}{6(x^2+x+1)}$$

$$= \frac{6x - 3(j+\bar{j}) + i\sqrt{3}(\bar{j}-j)}{6(x^2+x+1)} = F_1$$

or

$$\boxed{1+j+j^2=0} \quad \text{donc } j+\bar{j}=-1$$

$$j = -\frac{1+i\sqrt{3}}{2} ; \bar{j} = -\frac{1-i\sqrt{3}}{2} \Rightarrow \bar{j}-j = -i\sqrt{3}$$

$$\text{Donc } F_1 = \frac{6x+3+i\sqrt{3}(-i\sqrt{3})}{6(x^2+x+1)} = \frac{6x+3+3}{6(x^2+x+1)} = \frac{6x+6}{6(x^2+x+1)} = \frac{x+1}{x^2+x+1}$$

Donc la décomposition de f ds $\mathbb{R}(x)$ est:

$$F(x) = \frac{-x}{x^2+1} + \frac{x+1}{x^2+x+1}$$

2) Décomposer en éléments simples dans $\mathbb{C}(x)$ la fonction

$$F = \frac{x^5+1}{x(x-1)^2}$$

$$\text{On a : } F = E + \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2} \quad a, b, c \in \mathbb{C}$$

$$x(x-1)^2 = x(x^2-2x+1) = x^3-2x^2+x$$

$$\begin{array}{r} x^5+1 \\ x^3-2x^2+x \\ \hline x^2+2x+3 \end{array}$$

$$\begin{array}{r} 2x^2-4x+1 \\ 2x^3-4x^2+2x \\ \hline x^3-6x^2+3x \end{array}$$

$$\begin{array}{r} x^3-6x^2+3x \\ \hline 4x^2-3x+1 \end{array}$$

$$\text{Donc } E = x^2+2x+3$$

(3)

$$a = F(x) \times \left]_{x=0} = \frac{x^2+1}{(x-1)^2} \right]_{x=0} = \boxed{1}$$

$$c = F(x)(x-1)^2 \left]_{x=1} = \frac{x^2+1}{x} \right]_{x=1} = \boxed{2}$$

$$F = \frac{x^2+2x+3}{x} + \frac{1}{x} + \frac{b}{x-1} + \frac{2}{(x-1)^2}$$

$$f(-1) = 1 - 2 + 3 = 2 \quad -\frac{1}{2}b + \frac{2}{4}$$

$$0 = 1 + \frac{1}{2} - \frac{1}{2}b = \frac{3}{2} - \frac{1}{2}b \Rightarrow \boxed{b=3}$$

$$\text{Donc } F(x) = x^2 + 2x + 3 + \frac{1}{x} + \frac{3}{x-1} + \frac{2}{(x-1)^2}$$

3) Décomposer en éléments simples dans $\mathbb{R}(x)$

de fraction: $f = \frac{x+1}{(x-1)^2(x^2+1)^2}$ *hi faire / hi impaire*

$$\text{Donc } F = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2} \quad (a, b, c, d, e, f \in \mathbb{R})$$

$$\bullet F(x)(x-1)^2 \left]_{x=1} = b \Rightarrow b = \frac{x+1}{(x^2+1)^2} \right]_{x=1} = \boxed{\frac{1}{2}}$$

$$\bullet F(x) - \frac{1}{2(x-1)^2} = \frac{a}{x-1} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2}$$

$$G(x) = \frac{2(x+1) - (x^2+1)^2}{2(x-1)^2(x^2+1)^2} = \frac{a}{x-1} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2}$$

$$G(x) = \frac{2x+2 - x^4 - 2x^2 - 1}{2(x-1)^2(x^2+1)^2} = \frac{-x^4 - 2x^2 + 2x + 1}{2(x-1)^2(x^2+1)^2}$$

1 est racine de ce polynôme

$$\begin{array}{r|l}
 -x^4 - 2x^2 + 2x + 1 & x - 1 \\
 \hline
 -x^4 + x^3 & \\
 \hline
 -x^3 - 2x^2 + 2x + 1 & \\
 -x^3 + x^2 & \\
 \hline
 -3x^2 + 2x + 1 & \\
 -3x^2 + 3x & \\
 \hline
 -x + 1 & \\
 -x + 1 & \\
 \hline
 0 &
 \end{array}$$

Donc $G(x) = \frac{(x-1)(-x^3 - x^2 - 3x - 1)}{2(x-1)^2(x^2+1)^2} = \frac{-x^3 - x^2 - 3x - 1}{2(x-1)(x^2+1)^2}$

$$G(x) = \frac{a}{x-1} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2}$$

$$a = G(x)(x-1) \Big|_{x=1} = \frac{-x^3 - x^2 - 3x - 1}{2(x^2+1)^2} \Big|_{x=1} = \boxed{-\frac{3}{4}}$$

$$\begin{aligned}
 F(x)(x^2+1)^2 &= \left[\frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+1} \right] (x^2+1)^2 + (ex+f) \\
 &= \left(\frac{a}{x-1} + \frac{b}{(x-1)^2} \right) (x^2+1)^2 + (cx+d)(x^2+1) + ex + f
 \end{aligned}$$

$$\begin{aligned}
 F(x)(x^2+1)^2 \Big|_{x=i} &= \boxed{ei+f} \Rightarrow \frac{i+1}{(i-1)^2} = ei+f \Rightarrow \frac{i+1}{-2i} = ei+f \\
 \Rightarrow \frac{i^2+i}{+2} = ei+f &\Rightarrow \boxed{-\frac{1}{2} + \frac{i}{2} = ei+f} \Rightarrow \boxed{e = \frac{1}{2}}, \boxed{f = -\frac{1}{2}}
 \end{aligned}$$

$$F(x) = \frac{x+1}{(x-1)^2(x^2+1)^2} \Rightarrow \lim_{x \rightarrow \infty} x f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{x^6} = 0 \quad \lim_{x \rightarrow \infty} \frac{1}{x^4} = 0$$

Donc part: $F(x) = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+1} + \frac{ex+f}{(x^2+1)^2}$

$$\Rightarrow x f(x) = \frac{ax}{x-1} + \frac{bx}{(x-1)^2} + \frac{cx^2+dx}{x^2+1} + \frac{cx^2+fx}{(x^2+1)^2}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x f(x) = a + 0 + c + 0 \quad \text{Donc } 0 = a + c \Rightarrow \boxed{c = -a = \frac{3}{4}}$$

$$\begin{aligned}
 1 = F(0) &= -a + b + d + f \Rightarrow 1 = \frac{3}{4} + \frac{1}{2} + d + (-\frac{1}{2}) \Rightarrow \boxed{d = \frac{1-3}{4} = -\frac{1}{4}} \\
 \text{Donc } F(x) &= \frac{-3}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{\frac{3}{4}x + \frac{1}{4}}{x^2+1} + \frac{1}{2} \frac{(x-1)}{(x^2+1)^2}
 \end{aligned}$$