

Série d'exercices d'Algèbre n°1

Exercice 1 1. Soient $z_1 = 1 + i$, $z_2 = \sqrt{3} + i$, $z_3 = \frac{z_1}{z_2}$.

Déterminer le module et un argument des nombres complexes z_1 , z_2 et z_3 . En déduire les valeurs exactes de $\cos \frac{\pi}{12}$ et $\sin \frac{\pi}{12}$.

2. Comment choisir l'entier naturel n pour que $(\sqrt{3} + i)^n$ soit réel? Imaginaire?

Exercice 2 Déterminer les complexes z tels que z , $\frac{1}{z}$ et $1 + z$ aient même module.

Exercice 3 On note $j = \exp(2i\pi/3)$.

1. Calculer j^{-1} , $\bar{j}$, $1 + j + j^2$ en fonction de j .

2. Calculer j^n pour $n \in \mathbb{N}$.

3. Simplifier $j(1 + j)$, $\frac{j}{1+j^2}$, $\frac{1}{1+j}$ et $\frac{j+1}{j-1}$.

Exercice 4 1. Donner les solutions complexes de

$$X^4 - X^2 + 1 = 0, \quad X^6 = 1$$

2. Résoudre alors dans $\mathbb{C}$, l'équation suivante

$$X^6 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Exercice 5 Soit n un entier naturel supérieur ou égal à 2. On pose $\omega = \exp(i\frac{\pi}{n})$.

1. Justifier que $\omega \neq 1$.

2. On pose $S_n = \sum_{k=0}^{n-1} \omega^k$, montrer que

$$S_n = \frac{2}{1 - \omega}.$$

3. On considère les sommes

$$U_n = \sum_{k=0}^{n-1} \cos\left(\frac{k\pi}{n}\right) \quad \text{et} \quad V_n = \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{n}\right).$$

(a) Montrer que

$$U_n = 1 \quad \text{et} \quad V_n = \frac{1}{\tan(\pi/2n)}.$$

(b) En déduire que $\tan(\pi/8) = \sqrt{2} - 1$ puis que

$$\cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}} \quad \text{et} \quad \sin \frac{\pi}{8} = \frac{1}{2} \sqrt{2 - \sqrt{2}}.$$

(On pourra utiliser la formule $\cos^2 x = (1 + \tan^2 x)^{-1}$, pour $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$).

4. Calculer $T_n = \sum_{k=0}^{n-1} |\omega^{2k} - 1|$.

Ex 1

$$1) |z_1| = |1+i| = \sqrt{1^2+1^2} = \sqrt{2} \rightarrow z_1 = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$|z_2| = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2+1} = 2 \rightarrow z_2 = 2 \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = 2 e^{i\frac{\pi}{6}}$$

$$|z_3| = \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{2}}{2} \quad z_3 = \frac{z_1}{z_2} = \frac{\sqrt{2} e^{i\frac{\pi}{4}}}{2 e^{i\frac{\pi}{6}}} = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{12}}$$

$$2) z_3 = \frac{1+i}{\sqrt{3}+i} = \frac{(1+i)(\sqrt{3}-i)}{4} = \frac{\sqrt{3}-i+i\sqrt{3}+1}{4} = \frac{\sqrt{3}+1}{4} + i \frac{\sqrt{3}-1}{4}$$

$$= \frac{\sqrt{2}}{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$\Rightarrow \frac{\sqrt{3}+1}{4} = \frac{\sqrt{2}}{2} \cos \frac{\pi}{12} \rightarrow \cos \frac{\pi}{12} = \frac{\frac{\sqrt{3}+1}{4}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\Rightarrow \frac{\sqrt{3}-1}{4} = \frac{\sqrt{2}}{2} \sin \frac{\pi}{12} \Rightarrow \sin \frac{\pi}{12} = \frac{\frac{\sqrt{3}-1}{4}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$2) (\sqrt{3}+i)^n = z_2^n = \left[2 e^{i\frac{\pi}{6}} \right]^n = 2^n \left(\cos n\left(\frac{\pi}{6}\right) + i \sin n\left(\frac{\pi}{6}\right) \right)$$

$$\cdot z_2^n \text{ est réel } \Leftrightarrow \sin n\frac{\pi}{6} = 0 \Leftrightarrow n\frac{\pi}{6} = k\pi \quad ; k \in \mathbb{N}$$

$$\Leftrightarrow n = 6k \quad ; k \in \mathbb{N}$$

$$\cdot z_2^n \text{ imaginaire } \Leftrightarrow \cos n\frac{\pi}{6} = 0 \Leftrightarrow n\frac{\pi}{6} = \frac{\pi}{2} + k\pi \quad ; k \in \mathbb{N}$$

$$n = 3 + 6k \quad ; k \in \mathbb{N}$$

$$\text{Ex 2: } \begin{cases} |z| = \left| \frac{1}{z} \right| \\ |z| = |z+1| \end{cases} \Leftrightarrow \begin{cases} |z|^2 = 1 \\ |z|^2 = |z+1|^2 \end{cases} \Leftrightarrow \begin{cases} \operatorname{Im}(z)^2 + \operatorname{Re}(z)^2 = 1 \\ z\bar{z} = (1+z)(1+\bar{z}) \end{cases}$$

$$\Leftrightarrow \begin{cases} \operatorname{Im}^2(z) + \operatorname{Re}^2(z) = 1 \\ z\bar{z} = 1 + z\bar{z} + z + \bar{z} \end{cases}$$

$$\Leftrightarrow \begin{cases} \operatorname{Im}^2(z) + \operatorname{Re}^2(z) = 1 \\ 2 \operatorname{Re}(z) = -1 \end{cases}$$

$\Leftrightarrow$

$$\Leftrightarrow \begin{cases} \operatorname{Im}(z) = \frac{3}{4} \\ \operatorname{Re}(z) = \frac{1}{2} \end{cases}$$

$$\operatorname{Im}(z) = \frac{\sqrt{3}}{2} \text{ ou } \operatorname{Im}(z) = -\frac{\sqrt{3}}{2}$$

$$\operatorname{Re}(z) = -\frac{1}{2}$$

$$\Rightarrow z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ ou } z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$= j \text{ ou } \bar{j}$$

$$S_4 = \{j, \bar{j}\}$$

ou bien

$$|z| = \frac{1}{|z|} \Rightarrow |z| = 1$$

$$z \in E(0, 0, 1)$$

$$|1+z| = |z| \Leftrightarrow |z - (-1)| = |z - 0|$$

$$\Leftrightarrow AM = OM \rightarrow M \in \text{med}(\overline{OA}) \quad \text{avec } A(-1, 0) \quad \text{et } M = -\frac{1}{2}$$

$$\text{Ex 3: } j = e^{2i\frac{\pi}{3}} = e^{2i\frac{K\pi}{n}}$$

A retenir

$$j = e^{2i\frac{\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$j^3 = 1$$

$$\frac{1}{j} = \frac{1}{j} = j^2$$

$$1) \frac{1}{j^{2j^2}} = \frac{j^2}{j^3} = j^2$$

$$\frac{1}{j} = e^{-2i\frac{\pi}{3}} = e^{-\frac{2i\pi}{3} + 2i\pi} = e^{\frac{4i\pi}{3}} = (e^{2i\frac{\pi}{3}})^2 = j^2$$

qu'on peut passer d'un arg $\theta \rightarrow \theta + 2\pi$
 $\theta \rightarrow \theta - 2\pi$

$$1 + j + j^2 = 0?$$

$$\text{car } j^3 = 1$$

$$j^3 - 1 = 0 \text{ eq } (j-1)(1+j+j^2) = 0$$

$$\text{or } j-1 \neq 0 \text{ donc } j^2 + j + 1 = 0$$

$$j^2 + j + 1 = \bar{j} + j + 1 = 0$$

$$= 1 + 2\operatorname{Re}(j) = 1 - 1 = 0$$

2)

$$1 + j + j^2 = \sum_{k=0}^{n-1} j^k : \text{Somme d'une suite géométrique de ratio } j$$

$$= \frac{1-j^3}{1-j} = 0$$

$$a^n - b^n = ?$$

$$1 - x^n$$

$$= \sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$$

$$\text{donc } 1-x^n = (1-x) \left(\sum_{k=0}^{n-1} x^k \right)$$

$$a^n - b^n = a^n \left(1 - \left(\frac{b}{a} \right)^n \right) = a^n (1-b) \left(\sum_{k=0}^{n-1} \left(\frac{b}{a} \right)^k \right)$$

$$2) j^n = e^{\frac{2in\pi}{3}}$$

1er cas

$$n = 3k \\ j^n = j^{3k} = (j^3)^k = 1$$

$$n = 3k+1$$

$$j^n = j^{3k+1} = j^{3k} \cdot j = j$$

$$n = 3k+2$$

$$j^n = j^{3k+2} = j^{3k} \cdot j^2 = j^2$$

$$n \in \mathbb{N} \begin{cases} \rightarrow 3k \\ \rightarrow 3k+1 \\ \rightarrow 3k+2 \end{cases}$$

$$j = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$j^3 = 1$$

$$\bar{j} = j^2 = \frac{1}{j}$$

$$1+j+j^2=0$$

$$0 \quad \pi \quad \frac{\pi}{2} \quad \frac{\pi}{3} \quad \frac{\pi}{4} \quad \frac{\pi}{6}$$

$$e^{i\theta} = e^{i\theta} \cdot e^{i\theta} = e^{i2\theta} \quad (e^{i\theta})^2 = e^{i2\theta}$$

$$3) \cdot j(1+j) = j+j^2 = -1$$

$$\cdot \frac{j}{1+j^2} = \frac{j}{-j} = -1$$

$$3\bar{j} = |3|2$$

$$a |j-1|^2 = \left| -\frac{3}{2} + i\frac{\sqrt{3}}{2} \right|^2 \\ = 3$$

$$\cdot \frac{1}{1+j} = \frac{-1}{j^2 \cdot j} = \frac{-j}{j^3} = -j$$

$$\cdot \frac{j+1}{j-1} = \frac{-j^2}{1-j^2}$$

$$= \frac{(j+1)(\bar{j}-1)}{(j-1)(\bar{j}-1)} = \frac{1/j + j^2 - 1}{1-j-1+j^2}$$

$$= \frac{-j+j^2}{3} = \frac{1}{3} (\bar{j}-j)$$

$$= \frac{1}{3} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \right) = -\frac{i\sqrt{3}}{3}$$

Ex 4:

$$X^4 - X^2 + 1 = 0 \quad e_1 \quad y^2 - y + 1 = 0$$

$$\Delta = 1-4 = -3$$

$$\Delta = (i\sqrt{3})^2$$

$$\text{on pose } z = X^2$$

$$\text{ou } y_2 = \frac{1+i\sqrt{3}}{2}$$

$$e_1 \quad X^2 - X + 1 = 0$$

$$\Leftrightarrow y_1 = \frac{1-i\sqrt{3}}{2}$$

$$\text{ou } y_2 = e^{i\pi/3}$$

$$X =$$

$$\Leftrightarrow y_1 = e^{-i\pi/3}$$

$$= \bar{j}$$

$$z_3 = e^{i\pi/6}$$

$$-1 = e^{i\pi}$$

$$\Rightarrow z_1 = e^{-i\pi/6}$$

$$\text{ou } z_2 = e^{-i\pi/6} = e^{i5\pi/6}$$

$$z_4 = e^{i\pi/6} = e^{i7\pi/6}$$

$$S_4 = \{ e^{i\pi/6}, e^{-i\pi/6}, e^{i5\pi/6}, e^{i7\pi/6} \}$$

• $X^6 = 1$ les racines 6^{ème} de l'unité

autre exp

$$x = e^{\frac{2ik\pi}{6}} \quad k \in \{0, \dots, 5\}.$$

$$x^3 = 2 \quad \text{eq} \quad \frac{x}{2}$$

$$\Leftrightarrow x = e^{\frac{2k\pi}{3}} \quad k \in \{0, \dots, 5\}.$$

$$\text{eq} \quad \left(\frac{-}{3}\right)$$

$$2) \quad x^6 = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = -j = j^2 = e^{4i\pi/3}$$

l'equation

$$x^6 = e^{4i\pi/3}$$

$$\text{autre exp} \quad x^5 = (-$$

$$\text{eq} \quad \frac{x^6}{e^{4i\pi/3}} = 1$$

$$\text{eq} \quad \left(\frac{x}{e^{\frac{4i\pi}{3 \times 6}}} \right)^6 = 1 \quad \text{eq} \quad \left(\frac{x}{e^{\frac{2i\pi}{9}}} \right)^6 = 1$$

$$\text{eq} \quad (Y)^6 = 1$$

$$\text{eq} \quad \frac{x}{e^{\frac{2i\pi}{9}}} = e^{\frac{2ik\pi}{6}} \quad k \in \{0, \dots, 5\}.$$

$$\text{eq} \quad x = e^{\frac{2i\pi}{9}} \times e^{\frac{2ik\pi}{6}} \quad \text{avec } k \in \{0, \dots, 5\}.$$

cas:

$$w = \exp(i \frac{\pi}{n})$$

$$e^{i\theta} = e^{i\theta'}$$

$$\text{eq } \theta = \theta' + 2k\pi$$

1) $w \neq 1$ (raisonnement par récurrence)

On suppose que $w \neq 1$

$$\Leftrightarrow e^{i \frac{\pi}{n}} = 1 = e^{2i\pi} \quad \left(\frac{\pi}{n} = 2\pi + 2k\pi \right)$$

$$= 2\pi(1+k)$$

$$\Leftrightarrow e^{i \frac{\pi}{n}} = e^{2i\pi} \quad (k \in \mathbb{Z})$$

$$= 2\pi(1+k)$$

$$\Leftrightarrow \frac{\pi}{n} = 2k\pi \quad (k \in \mathbb{Z})$$

$$\text{eq } m = \frac{1}{2k} \quad (k \in \mathbb{Z})$$

$$n \notin \mathbb{N} \quad (\text{absurde})$$

donc $w \neq 1$

$$2) S_n = \sum_{k=0}^{n-1} w^k$$

somme d'une suite géométrique

de raison w et de premier terme $w^0 = 1$

$$= \frac{1 - w^n}{1 - w} \times 1 = \frac{1 - e^{i\pi}}{1 - w} = \frac{2}{1 - w}$$

Rq: $z_n = a_n + i b_n$: suite complexe.
 $\underbrace{a_n}_{\text{réel}} \underbrace{+ i}_{\text{réel}} \underbrace{b_n}_{\text{réel}} \rightarrow$ toutes les règles des
 suites réelles sont valables.

sauf que pas de comparaison.

$$3) \sum_{k=0}^{n-1} i V_k = \sum_{k=0}^{n-1} \left(\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n} \right)$$

$$= \sum_{k=0}^{n-1} e^{i \frac{k\pi}{n}}$$

$$= \sum_{k=0}^{n-1} w^k = \frac{2}{1 - w}$$

$$= \frac{2}{1 - e^{i \frac{\pi}{n}}} = \frac{2}{e^{i \frac{\pi}{2n}} (e^{-i \frac{\pi}{2n}} - e^{i \frac{\pi}{2n}})}$$

$$= \frac{2 e^{-i \frac{\pi}{2n}}}{-2i \sin(\frac{\pi}{2n})} = \frac{i e^{-i \frac{\pi}{2n}}}{\sin(\frac{\pi}{2n})}$$

$$= \frac{i (\cos(\frac{\pi}{2n}) - i \sin \frac{\pi}{2n})}{\sin(\frac{\pi}{2n})}$$

$$= 1 + i \frac{\cos(\frac{\pi}{2n})}{\sin(\frac{\pi}{2n})} = 1 + i \frac{1}{\tan(\frac{\pi}{2n})}$$

$$\text{eq } u_n = 1 \quad \text{et} \quad v_n = \frac{1}{\tan(\frac{\pi}{2n})}$$

$$b) v_n = \frac{1}{\tan(\frac{\pi}{2n})}$$

$$\text{pour } n=4 \quad \frac{1}{\tan(\frac{\pi}{8})} = v_4 = \sum_{k=0}^3 \sin \frac{k\pi}{4}$$

$$= 0 + \sin \frac{\pi}{4} + \sin \frac{\pi}{2} + \sin \frac{3\pi}{4}$$

$$= \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} = \sqrt{2} + 1$$

$$\text{eq } \tan(\frac{\pi}{8}) = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$$

$$\cos^2(\frac{\pi}{8}) = \frac{1}{1 + \tan^2(\frac{\pi}{8})} = \frac{1}{1 + (\sqrt{2} - 1)^2}$$

$$= \frac{1}{4 - 2\sqrt{2}}$$

$$= \frac{4 + 2\sqrt{2}}{8} = \frac{2 + \sqrt{2}}{4}$$

$$\sqrt{x^2} = |x|$$

$$\Rightarrow \left| \cos(\frac{\pi}{8}) \right| = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\text{or } \frac{\pi}{8} \in [0, \frac{\pi}{2}]$$

$$\Rightarrow \cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\cos \frac{\pi}{8} > 0$$

$$\sin(\frac{\pi}{8}) = \cos(\frac{\pi}{8}) \times \tan(\frac{\pi}{8})$$

$$= (\sqrt{2} - 1) \times \left(\frac{1}{2} \sqrt{2 + \sqrt{2}} \right)$$

$$= \frac{1}{2} (\sqrt{2} - 1) (\sqrt{2 + \sqrt{2}}) = \frac{1}{2} \sqrt{2} - \sqrt{2}$$

$$= \frac{1}{2} (\sqrt{2} - 1) \sqrt{2 + \sqrt{2}} = \rightarrow$$

4)

$$T_n = \sum_{k=0}^{n-1} |w^{2k} - 1|$$

Rq $e^{i\theta} -$
 $|e^{i\theta}| = 1$
 si θ réel

$$= \sum_{k=0}^{n-1} \left| e^{\frac{2i k \pi}{n}} - 1 \right|$$

$$= \sum_{k=0}^{n-1} \left| e^{\frac{i k \pi}{n}} (e^{\frac{i k \pi}{n}} - e^{-\frac{i k \pi}{n}}) \right|$$

$$= \sum_{k=0}^{n-1} \underbrace{\left| e^{\frac{i k \pi}{n}} \right|}_1 \left| e^{\frac{i k \pi}{n}} - e^{-\frac{i k \pi}{n}} \right|$$

$$= \sum_{k=0}^{n-1} \left| 2i \sin \frac{k\pi}{n} \right|$$

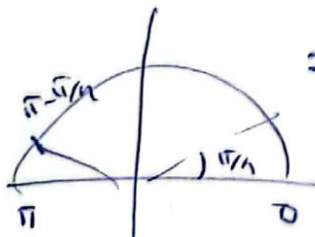
$$= 2 \sum_{k=0}^{n-1} \left| \sin \frac{k\pi}{n} \right|$$

(*)

$$0 \leq k \leq n-1 \quad = 2 \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{n}\right)$$

$$0 \leq k\pi \leq \pi n - \pi$$

$$0 \leq \frac{k\pi}{n} \leq n - \frac{\pi}{n} \quad \textcircled{**} \quad = 2 \forall n = \frac{2}{\tan\left(\frac{\pi}{2n}\right)}$$



$$\Rightarrow \sin\left(\frac{k\pi}{n}\right) \geq 0$$